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Plane Wave Solutions of a Quantum Fractional Schrödinger Equation and Uncertainty Principle

Muhammad I. Bhatti
Department of Physics, University of Texas Rio Grande Valley, Edinburg, USA
muhammad.bhatti@utrgv.edu

Abstract:— **Plane wave solutions of the fully fractional Schrödinger equation were proposed and represented in terms of exponential function. The plane wave solutions satisfied the fractional time-dependent Schrödinger equation. The Uncertainty Principle was obtained from the solution in the one-dimensional case using α as a fractional order parameter of the space and time derivatives. For the integral value of the fractional parameter α, the standard solution of the Schrödinger Equation was recovered. Some physical quantities such as the Mean Square Distance and expectation of the fractional momentum were evaluated. For the integral value of α=1 the expressions of these physical quantities returned to standard quantum mechanical formulae.**

Keyword:— **Fractional Schrödinger Equation, Fractional Derivatives, Fractional Integration, Fractional time-dependent Equation**

1. Introduction

Applying partial fractional equations to the diffusion equation has made it possible to describe complicated systems with strange behavior in much the same way as simpler systems. Numerous authors [1-10] have studied many examples of a fractional harmonic oscillator, homogeneous fractional ordinary differential equations, a fractional diffusion equation and a wave equation. Solutions of fractional order homogeneous and non-homogenous partial differential equations and integral equations have also been considered [7-9]. A large body of scientific work has been published using the methods of fractional calculus to study quantum phenomena [11-15]. The Schrödinger equation has been generalized using partial fractional equations in many ways, including, (a) keeping the first order time derivative and generalizing the fractional space derivative [14, 16] and (b) including the fractional order of the time derivative but retaining the space derivatives [12, 13]. There also exist more generalized forms of the Schrödinger equation where both space and time derivatives of fractional order are considered [17]. In these studies, fractional derivatives of the Caputo or Riemann-Liouville types were employed. In works [14, 16], the space-time Schrödinger equation (STSE) was derived using the Feynman path integral technique and Levey-like quantum paths. However, the STSE was still first order in the time derivative, and only the space derivatives were extended to fractional-order. Parity conservation and the current density were explored using Riesz's fractional derivative. Applications of STSE have covered the dynamics of a free particle in the infinite well, a fractional Bohr atom and a quantum fractional oscillator [16]. In work [13], the fractional time derivative was introduced by replacing the time derivative and the imaginary number with a derivative of fractional order, leaving the space derivative intact in the Schrödinger equation. More recently [12], a fractional time-dependent Schrödinger equation was derived using the Feynman path integral technique, showing that there was no need to raise the power of the imaginary number *i* to fractional-order, while leaving the space derivative intact.

In the present work, modifications to the Schrödinger equation were considered in both time and space fractional-order derivatives, based on work [11]. A fractional plane wave solution was considered, which satisfied the fully generalized fractional-order Schrödinger Equation (FSE); also, the generalized Uncertainty Principle was verified as a check. In addition, the Mean Square Distance (MSD) was calculated using the proposed fractional solution. In this paper, we were concerned with fractional derivatives of order α as the fractional parameter for the space and time derivatives of the FSE. The physical significance of the parameter α is that the variations are coarse-grained in the space-time, as mentioned in [18, 19].

2. Fractional Wave Equation

Consider a Fractional Schrödinger Equation (FSE) of the form,

$$i\hbar^{\alpha} D_t^{\alpha}\psi(\mathbf{x},\mathrm{t}) = B_{\alpha}(-D_x^{2\alpha})\psi(\mathbf{x},\mathrm{t}),\quad \frac{1}{2}<\alpha\le 1, \tag{2.1}$$

where the fractional derivative of order α is $D_x^{\alpha}=\dfrac{\partial^{\alpha}}{\partial x^{\alpha}}$, $\hbar$ is the reduced Planck's constant, m is the mass of the particle and $\psi(\mathbf{x},\mathrm{t})$ is a wave function associated with a free particle. Also, we set constants $B_{\alpha}=\left(\dfrac{\hbar^2}{2m}\right)^{\alpha}$ with units of $\mathrm{j}^{\alpha}\square m^{2\alpha}$ and $b_{\alpha}=\left(\dfrac{\hbar}{2m}\right)^{\alpha}$

with units of $m^{2\alpha}.s^{-\alpha}$, respectively. In the event $\alpha \to 1$, these constants return to standard constants. Essentially, Eq. (2.1) is obtained by replacing the momentum operator with $p^{\alpha} \to -i\hbar^{\alpha} D_x^{\alpha}$ and the energy operator with $E^{\alpha} \to i\hbar^{\alpha} D_t^{\alpha}$ in the equation $E^{\alpha} = \frac{p^{2\alpha}}{(2m)^{\alpha}}$. We solve this equation in 1-D with the following boundary conditions:

$$\begin{aligned} &\psi(x,0)=\psi_0(x) \\ &\psi(x,t) \to 0 \quad \text{as } |x| \to \infty, t>0. \end{aligned} \tag{2.2}$$

In the literature, we found several fractional calculus formulae (see [18] and [19]) which for the sake of brevity are listed below:

1. Fractional Integrals with respect to $(dx)^{\alpha}$

$$\begin{cases} I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{-\alpha} f(\xi)\, d\xi \\ I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)} \int_0^x f(\xi)(d\xi)^{\alpha} \quad 0<\alpha\le 1 \\ \int (dx)^{\alpha} = x^{\alpha}/\Gamma(\alpha+1) \end{cases} \tag{2.3}$$

2. Some fractional derivative formulae:

$$\begin{cases} D_x^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} \\ D_x^{\alpha}\left(u(x)v(x)\right) = \left[D_x^{\alpha} u(x)\right] v(x) + u(x)\left[D_x^{\alpha} v(x)\right] \\ D_x^{\alpha}\left(f(u(x)\right) = f_u'\left(u(x)\right) D_x^{\alpha} u(x) = D_x^{\alpha} f\left(u(x)\right)\left(u'(x)\right)^{\alpha} \\ D_x^{\alpha}\left(x^{\alpha}\right)^2 = 2(\alpha!)\, x^{\alpha} \end{cases} \tag{2.4}$$

The normalized solution that satisfied the FSE, Eq. (2.1), in terms of space and time may be written in the form:

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left(\frac{a^{\alpha}}{\sqrt{\pi}\,\Gamma(\alpha+1)} \right)^{\frac{1}{2}} \frac{e^{-\frac{\left(|x|^{\alpha}\right)^2}{4\Gamma^2(\alpha+1)\,\beta^2(t)}}}{\beta(t)}, \tag{2.5}$$

where the beta function is defined as

$$\beta^2(t) = \left(\frac{a^{2\alpha}}{2\Gamma^2(\alpha+1)} + \frac{i\, b_{\alpha} t^{\alpha}}{\Gamma(\alpha+1)} \right). \tag{2.6}$$

The constant a represents an initial Gaussian width of the initial wave. Note that the wave function given in Eq. (2.5) satisfies the fractional Schrödinger equation (2.1). This can be verified using the fractional derivatives provided in Eq. (2.4). Also, using the wave function in Eq. (2.5), we would obtain the quantum mechanical probability density by taking the complex conjugate of the wave function and multiplying it by the wave function itself, $P(x,t) = \Psi^{*}\Psi$. The expression for the probability density is:

$$P(x,t) = \left(\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\, a^{\alpha}\, c} \right) e^{\frac{\left(|x|^{\alpha}\right)^2}{a^{2\alpha} c^2(t)}}. \tag{2.7}$$

The function c(t) in the above probability function is given below:

$$c^2(t) = \left(1 + \frac{4\Gamma^2(\alpha+1) b_{\alpha}^2 t^{2\alpha}}{a^{4\alpha}} \right). \tag{2.8}$$

Note that as $\alpha \to 1$ the expressions in Eqs. (2.5-2.8) returns to the standard quantum mechanical expressions [11].

3. Fractional Uncertainty Principle

An important physical quantity, the Fractional Mean Square Distance (MSD), may be calculated using the quantum mechanical probability in Eq. (2.7) and integrating with respect to $(dx)^{\alpha}$ using the formulae in Eq. (2.3), which yields:

$$\left\langle x^{2\alpha} \right\rangle = \frac{1}{2} a^{2\alpha} c^2. \tag{3.1}$$

The square root of Eq. (3.1) provides an expression for uncertainty in the position.

$$(\Delta x)_{\alpha} = \frac{1}{\sqrt{2}} a^{\alpha} c. \tag{3.2}$$

The above expression in Eq. (3.2) as $\alpha \to 1$ reduced to the standard result for uncertainty in the position x. It is straightforward to calculate the expectation value of the Fractional Momentum Squared with a fractional-order of α and carry out fractional integration, yielding:

$$\left\langle (\hbar k)^{2\alpha} \right\rangle = \frac{\hbar^{2\alpha}}{2} \frac{\Gamma^2(\alpha+1)}{a^{2\alpha}}. \tag{3.3}$$

The factor gamma squared can be absorbed in the momentum. Here, a normalized initial Gaussian wave function of the form has been used in the integration

$$\psi(k,0) = \left(\frac{a^{\alpha}}{\sqrt{\pi}} \right)^{\frac{1}{2}} e^{-\frac{\left(|k|^{\alpha} a^{\alpha}\right)^2}{2\Gamma^2(\alpha+1)}} \tag{3.4}$$

Taking the square root of Eq. (3.3), one would get uncertainty in the momentum,

$$(\Delta p)_{\alpha} = \frac{1}{\sqrt{2}} \left(\frac{\hbar}{a} \right)^{\alpha} \Gamma(\alpha+1). \tag{3.5}$$

The fractional uncertainty principle is just the product of both expressions given in equations (3.2) and (3.5),

$$(\Delta x \Delta p)_{\alpha} \ge \frac{\hbar^{\alpha}}{2} \Gamma(\alpha+1)\, c, \;\; 0<\alpha\le 1. \tag{3.6}$$

The expression of c(t) is provided in Eq. (2.8). The above expression with t = 0 and $\alpha \to 1$ reduces to the well-known Heisenberg Uncertainty Principle given by $\Delta x \Delta p \geq \frac{\hbar}{2}$. The new expression in Eq. (3.6) is a generalized fractional uncertainty principle. Here we varied the value of $\alpha = 2/3, 3/4, 0.85, 1$ and we get new expressions for uncertainty values that are greater than $\hbar/2$ for $\alpha < 1$. The above evaluations show that fractional calculus is a powerful tool that can be used to solve general problems and the reduction leads to the known standard quantum mechanical results when integral values of fractional order α are used.

4. Conclusion and Remarks

We have considered the Fractional Schrödinger equation (FSE) and found its Plane wave solution that satisfied the equation (2.1), and with the help of the fractional derivatives provided in Eq. (2.4), it can be verified. The proposed solution satisfies the fractional uncertainty principle; for the integral value of $\alpha = 1$, the FSE and the uncertainty principle were recovered as found in standard quantum mechanics. Applications of the time-independent fractional Schrödinger Equation will be considered in a forthcoming publication. Several new formulae are presented for various quantum mechanical quantities such as the fractional mean square distance and the uncertainty principle. More detailed properties of the fractional wave equation and its possible applications to the physical problems will be considered in future research.

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