

Batch Arrival $M^X/G/1$ Retrial Queueing System With General Retrial Time, Bernoulli Schedule Vacation, Multi Second Optional Heterogeneous Service And Repair Facility

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Abstract - This paper is concerned with the analysis of a single server batch general retrial queue with Bernoulli schedule vacation with server breakdowns. The server provides two phases of heterogeneous service where first service is essential and second service is multioptional. The steady state solutions are obtained in a closed form using Supplementary variable technique. Reliability measures and special cases are discussed. 2000 Mathematics subject classification: 60K25.

Keywords - Bernoulli Schedule vacation, $M^X/G/1$ queue, retrial queues, supplementary variable technique.

1. INTRODUCTION

In classical Queueing theory, it is assumed that any customer who cannot get service immediately upon arrival, either joins a waiting line or leaves the system forever. But there are real situations where the blocked customers leave the service area temporarily but returns to repeat their demand after some random time. This Queueing behavior is referred as retrial queues. Retrial Queueing systems are characterized by the fact that a primary request, finding all servers and waiting positions busy upon arrival, leaves the service area but after some random time, repeats its demand. Between retrials, the customer is said to be in "Orbit". So the repeated attempts for service from the group of blocked customers are super imposed on the normal stream of arrivals of primary requests. Thus the retrial queues can be considered as alternative to queues with losses that do not take repeated attempts into account. Retrial queues have wide applications in telephone systems, computer and communication networks, random access protocols in digital communication networks and daily life situations. For a review of main results and methods, the reader is referred to the specific monographs by Falin and Templeton (1997).

Recently there have been several contributions considering Queueing systems in which the server provides a First Essential Service (FES) to all the arriving customers, whereas only some of them receive a Second Optional Service (SOS). Krishna Kumar et al. (2002)

and Choudhury (2008) initially imposed the concept of "retrial customers" in the two-phase service models. Later several authors including Senthil Kumar and Arumuganathan (2008, 2010), Choudhury and Deka (2009) have analyzed two-phase service retrial Queueing models combined with server's vacations, N-policy, unreliable server, starting failure with feedback, etc. The above mentioned papers together with their references reveal that no work has been done in retrial queues with second multi-optional service taking into account server failures and repairs. Li and Wang (2006) studied a repairable $M/G/1$ retrial queue with FCFS orbit, general retrial time and second multi-optional services allowing balking of new arriving customers and reneging of customers in the retrial queues. It is assumed that the customer at the head of the retrial queue competes with the potential primary customer to decide which customer will enter service next.

Later Wang and Li (2008) analyzed a more general repairable $M/G/1$ retrial Queueing model with general retrial times, **Bernoulli schedule vacation** and second multi-optional service in which the server is allowed to take vacation after finishing a service to a customer. It is also assumed that, the server soon after the vacation period, must spend some time for setup operation before starting a new service. The steady state distributions of the orbit and system size are discussed under a steady state condition and some reliability characteristics such as availability and failure frequency of the server are also obtained.

In this paper, a batch arrival retrial Queueing model which generalizes the model of Wang and Li (2008) is considered. The Bernoulli Schedule Vacation is controlled by different parameters and the server who returns from vacation may spend random amount of time for setup work if necessary before starting the next service. The joint distributions of the server state and the queue length are compared with the model of Wang and Li (2008) under a corresponding stability condition and the mean queue length is also derived.

2 MATHEMATICAL ANALYSIS OF THE SYSTEM

2.1 Model Description

Customers arrive in batches at the system according to a time homogeneous Poisson process with group arrival rate λ . The batch size X is a random variable with probability distribution $\Pr(X = k) = g_k, k = 1, 2, \dots$ and

$$\sum_{k=1}^{\infty} g_k = 1.$$

There is no waiting space in front of the server, therefore, if an arriving batch finds the server idle, then one of the arriving customers begin to receive his service immediately and the others leave the service area and enter a group of blocked customers called “orbit” according to FCFS queue discipline. The customer at the head of the retrial queue competes with potential primary customers to decide which customer will enter the next service. If a batch of primary customers arrives first, the retrial customer may cancel its attempt for service and either returns to its position in the retrial queue with probability q or quits the system with probability $(1 - q)$. The retrial time of the customer in the retrial queue is generally distributed with distribution function $A(t)$, density function $a(t)$ and LST $A^*(\theta)$. Further it is assumed that, the retrial times begin only when the server is freely available in the system (i.e., either at the completion instants of services or setups or vacation completion instants). When the server is idle, an arriving (new arrival) customer must turn on the server immediately. If the server is found busy or on vacation or in breakdown state, the arriving batch joins the retrial queue according to FCFS discipline.

The server provides First Essential Service (FES) in the first phase and ‘c’ multi-optional services in the second stage. The First Essential Service (FES) is needed to all the arriving customers and the service times of FES follows a general distribution $S_0(t)$. As soon as the FES is completed, then with probability $r_i (1 \leq i \leq c)$, the customer may opt for the i^{th} type second service where the service will immediately commence, or else with

probability $r_0 = 1 - \sum_{i=1}^c r_i$, the customer may opt to leave

the system. The i^{th} type service time is assumed to follow arbitrary distribution $S_i(x), (1 \leq i \leq c)$.

The server’s lifetime is assumed to follow exponential distribution with mean $1/a_0$ in the FES. In the second multi-optional service, the server fails at an exponential rate $a_i (1 \leq i \leq c)$. When the server breaks down, it is sent for repair immediately and the customer just being served before server breaks down, waits in system for the server, to complete its remaining service. Immediately after the server is fixed, it starts to serve customers and the service time is cumulative. The repair time distributions of both service phases R_0 and $R_i (1 \leq i \leq c)$ are arbitrarily distributed with probability distribution functions $R_0(y)$ and $R_i(y) (1 \leq i \leq c)$ respectively.

After completing each service, the server is allowed to take a Bernoulli Schedule Vacation. If a customer leaves the system after completing FES, then the server may take vacation with probability p_0 or may remain in the

system to serve the next customer with probability $(1 - p_0)$. On the other hand, if the customer after finishing FES, opts for the i^{th} second optional service, then the server with probability p_i will take vacation after completing the i^{th} SOS to the customer or may stay in the system to serve the next customer with probability $(1 - p_i)$. The vacation time V in either case follows a general distribution $V(t)$. In either case, even if there is no customer in the system, the server adopts the same control policy i.e., either takes vacation or waits for the next customer to arrive or retry.

The server after returning from vacation may need time for preparatory work before starting a new service. This preparatory time corresponds to setup time D . The probability with which the server opts for the setup operation is b . If the server does not need any setup operation after returning from vacation, then the server directly joins the service facility with probability $(1 - b)$ and waits to serve the next customer.

The steady state system size equations under the steady state condition are analyzed by using supplementary variable technique. The PGF of the system size is obtained in a closed form so that various performance measures can be derived from it. The following usual notations are used to analyze this model.

- λ : Group arrival rate
- X : Group size random variable
- g_k : $\Pr(X = k), k = 1, 2, 3, \dots$
- $X(z)$: Probability generating function of X
- $N(t)$: The system size at time t

The notations of random variables (RV), Cumulative Distribution Functions (CDF), Probability Density Functions (PDF), Laplace-Stieltjes Transforms (LST) and the k^{th} moments of the random variables are listed below :

	RV	CDF	PDF	LST	k^{th} moments
Retrial time	A	$A(w)$	$a(w)$	$A^*(\theta)$	$E(A^k)$
Setup time	D	$D(x)$	$d(x)$	$D^*(\theta)$	$E(D^k)$
Vacation time	V	$V(x)$	$v(x)$	$V^*(\theta)$	$E(V^k)$
FES time	S_0	$S_0(x)$	$s_0(x)$	$S_0^*(\theta)$	$E(S_0^k)$
SOS time	S_i	$S_i(x)$	$s_i(x)$	$S_i^*(\theta)$	$E(S_i^k), 1 \leq i \leq c$
Repair time	R_i	$R_i(y)$	$r_i(y)$	$R_i^{*1}(\theta_1)$	$E(R_i^k), 0 \leq i \leq c$

Let $V^0(t), D^0(t), A^0(t), S_0^0(t), S_i^0(t), R_0^0(t)$ and $R_i^0(t), i = 1$ to c denote respectively the **remaining times** of the random variables : vacation time, setup time, retrial time, FES time, i^{th} type of SOS time, repair time due to FES and repair time due to the i^{th} type of SOS at time t . Further the different states of the server at time t are denoted by $Y(t) = \{0, 1, 2, 3, 4, 5, 6\}$ according as the server is on **vacation**, in **setup state**, in **idle state**, **busy with FES**, **busy with i^{th} type of SOS**, under **repair state due to FES** and **repair due to SOS** respectively. The supplementary variables are introduced in order to obtain

a Markov process $\{N(t), \delta(t)\}$ where $\delta(t) = (V^o(t), D^o(t), A^o(t), S_0^o(t), S_i^o(t), R_0^o(t), R_i^o(t), i = 1 \text{ to } c)$ according as $Y(t) = (0, 1, 2, 3, 4, 5, 6)$ respectively.

Let

$$Q_n(x, t) dt = \Pr(N(t) = n, x \leq V^o(t) \leq x + dt, Y(t) = 0), n \geq 0$$

$$D_n(x, t) dt = \Pr(N(t) = n, x \leq D^o(t) \leq x + dt, Y(t) = 1), n \geq 1$$

$$P_{0,1}(t) = \Pr(N(t) = 0, Y(t) = 2), n = 0$$

$$P_{n,1}(w, t) dt = \Pr(N(t) = n, w \leq A^o(t) \leq w + dt, Y(t) = 2), n \geq 1$$

$$P_{n,0}(x, t) = \Pr(N(t) = n, x \leq S_0^o(t) \leq x + dt, Y(t) = 3), n \geq 1$$

$$P_{n,i}(x, t) dt = \Pr(N(t) = n, x \leq S_i^o(t) \leq x + dt, Y(t) = 4), 1 \leq i \leq c, n \geq 1$$

$$B_{n,0}(x, y, t) dt = \Pr(N(t) = n, S_0^o(t) = x, y \leq R_0^o(t) \leq Y + dt, Y(t) = 5), n \geq 1$$

$$B_{n,j}(x, y, t) dt = \Pr(N(t) = n, S_j^o(t) = x, y \leq R_j^o(t) \leq Y + dt, Y(t) = 6), n \geq 1$$

Thus $Q_n(x, t) dt$ is the joint probability that at time t , there are n customers in the retrial orbit, and the remaining **vacation** time of the server is between x and $x + dt$, where $n \geq 0$.

$D_n(x, t) dt$ is the joint probability that at time t , there are n customers in the retrial orbit, and the remaining **setup** of the server is between x and $x + dt$, where $n \geq 1$.

$P_{n,1}(w, t) dt$ is the joint probability that at time t , there are n customers in the retrial orbit, the server is **idle**, and the remaining retrial time of the server is between w and $w + dt$, where $n \geq 1$ and $P_{0,1}$ is the probability that the server is idle at time t , and there is **no customer** in the retrial orbit.

$P_{n,i}(x, t) dt, i = 0 (1, \dots, c)$ is the joint probability that at time t , there are n customers in the retrial orbit, the server is busy, and a customer is being served in **FES** (i^{th} type of SOS) with remaining service time between x and $x + dt, n \geq 1$.

$B_{n,i}(x, t) dt, i = 0 (1, 2, \dots, c)$ is the joint probability that at time t , there are n customers in the retrial orbit, the remaining service time for the customer under the i^{th} type service is equal to x , and the server is being repaired with the remaining repair time between y and $y + dt$, where $x \neq 0, n \geq 0$.

And $Q_n(0), D_n(0), P_{n,1}(0), P_{n,i}(x, 0), B_{n,i}(x, 0), i = 0, 1, 2 \dots, c$, denote the probability that there are n customers in the retrial orbit at the termination of vacation period, setup period, idle period, service period and breakdown period respectively.

2.2 The System Size Distribution

Assuming the steady state probabilities exists as, $t \rightarrow \infty$ and independent of time t , the following steady state equations are obtained for the queueing system using supplementary variables technique :

$$\lambda P_{0,1} = (1 - p_0) r_0 P_{0,0}(0) + \sum_{i=1}^c (1 - p_i) P_{0,i}(0) + D_0(0) + (1 - b) Q_0(0)$$

$$\frac{-d}{dx} P_{n,1}(w) = -\lambda P_{n,1}(w) + (1 - p_0) r_0 P_{n,0}(0) a(w) + \sum_{i=1}^c (1 - p_i) P_{n,i}(0) a(w) + D_n(0) a(w) + (1 - b) Q_n(0) a(w), n \geq 1$$

$$\frac{-d}{dx} Q_n(x) = -\lambda Q_n(x) + p_0 r_0 P_{n,0}(0) v(x) + \sum_{i=1}^c p_i P_{n,i}(0) v(x) + \lambda \sum_{k=1}^n Q_{n-k}(x) g_k, n \geq 0$$

$$\frac{-d}{dx} D_n(x) = -\lambda D_n(x) + b Q_n(0) d(x) + \sum_{k=1}^n D_{n-k}(x) g_k, n \geq 0$$

$$\frac{-d}{dx} P_{0,0}(x) = -(\lambda + a_0) P_{0,0}(x) + P_{1,1}(0) s_0(x) + B_{0,0}(x, 0) + (1 - q) \lambda s_0(x) \left(\int_0^\infty P_{1,1}(w) dw g_1 \right) + \lambda P_{0,1} g_1 s_0(x)$$

$$\frac{-d}{dx} P_{n,0}(x) = -(\lambda + a_0) P_{n,0}(x) + P_{n+1,1}(0) s_0(x) + B_{n,0}(x, 0) + \lambda P_{0,1} g_{n+1} s_0(x)$$

$$+ \lambda \sum_{k=1}^n P_{n-k,0}(x) g_k + q \lambda s_0(x) \left(\int_0^\infty \sum_{k=1}^n P_{n-k+1,1}(w) dw g_k \right) + (1 - q) \lambda s_0(x) \left(\int_0^\infty \sum_{k=1}^{n+1} P_{n-k+2,1}(w) dw g_k \right), n \geq 1$$

$$\frac{-d}{dx} P_{n,i}(x) = -(\lambda + a_i) P_{n,i}(x) + r_i P_{n,0}(0) s_i(x) + B_{n,i}(x, 0) + \lambda \sum_{k=1}^n P_{n-k,i}(x) g_k, n \geq 0, 1 \leq i \leq c$$

$$\frac{-\partial}{\partial y} B_{n,0}(x, y) = -\lambda B_{n,0}(x, y) + a_0 P_{n,0}(x) r_0(y) + \lambda \sum_{k=1}^n B_{n-k,0}(x, y) g_k, n \geq 0$$

$$\frac{-\partial}{\partial y} B_{n,i}(x, y) = -\lambda B_{n,i}(x, y) + a_i P_{n,i}(x) r_i(y) + \lambda \sum_{k=1}^n B_{n-k,i}(x, y) g_k, n \geq 0, 1 \leq i \leq c$$

By algebraic manipulation, the total PGF is given by

$$P(z) = \lambda P_{0,1} \left[P_1(z) \psi(z) + \frac{A^*(\lambda)}{\lambda} \right]$$

$$\text{where } \psi(z) = \left[\frac{1 - S^*(w_x(z))}{w_x(z)} + \frac{(1 - A^*(\lambda))}{\lambda} S^*(w_x(z)) \right]$$

By using the normalizing condition $P_{0,1} = \frac{(1 - \rho - M_1'(1))}{d_R}$

$$\text{where } d_R = (1 - q) (1 - A^*(\lambda)) (\lambda E(S) + 1) + A^*(\lambda)$$

$$M_1'(1) = (1 - A^*(\lambda)) (E(X) + q - 1)$$

$$\rho = \lambda E(X) E(S)$$

$$E(S) = E(S_0)(1+a_0E(R_0))+ \sum_{i=1}^c r_i E(S_i)(1+a_i$$

$$E(R_i))+ \sum_{i=1}^c p_i r_i (E(V)+b E(D))$$

Following the arguments of Wang and Li (2008), it is found that $M_1'(1) = (1 - A^*(\lambda)) (E(X) + q - 1) < (1 - \rho)$ is the necessary and sufficient condition for the system to be stable.

2.3 Performance Measures

In this section, we present some system queueing measures as well as reliability indexes of the server under the steady state condition.

(i) The probability that the server is idle and system is empty is given by $P_{0,I} = \frac{(1-\rho-M_1'(1))}{d_R}$

(ii) The probability that the server is idle, but the system is not empty is given by $P_1 = \frac{(1-A^*(\lambda))(\rho+E(X)-1)}{d_R}$

(iii) The probability that the server is busy (P_{busy}) is given by $\frac{\lambda(E(S_0)+\sum_{i=1}^c r_i E(S_i))[(1-q)(1-A^*(\lambda))+A^*(\lambda)E(X)]}{d_R}$

(iv) The probability that the server (P_{br}) is under repair is given by $\frac{\lambda[E(S_0)a_0E(R_0)+\sum_{i=1}^c r_i E(S_i)a_iE(R_i)][(1-q)(1-A^*(\lambda))+A^*(\lambda)E(X)]}{d_R}$

(v) The probability that the server (P_V) is on vacation is given by $\frac{\lambda E(V)(\rho_0 r_0 + \sum_{i=1}^c p_i r_i)[(1-q)(1-A^*(\lambda))+A^*(\lambda)E(X)]}{d_R}$

(vi) The probability that the server is in setup period (P_{set}) is given by $\frac{\lambda b E(D) \sum_{i=0}^c p_i r_i [(1-q)(1-A^*(\lambda))+A^*(\lambda)E(X)]}{d_R}$

(vii) The probability that the server is not working, namely, the server is idle or under repair or on vacation or doing setup operation is given by $P = P_{0,I} + P_1 + P_{br} + P_V + P_{set}$

$$= \{[(1-q)(1-A^*(\lambda))+A^*(\lambda)E(X)]$$

$$[\lambda(E(V)+bE(D))(\rho_0 r_0 + \sum_{i=1}^c p_i r_i)$$

$$+ \lambda E(X)(E(S_0)a_0E(R_0) + \sum_{i=1}^c a_i E(R_i)E(S_i))]$$

$$+ (1-q+qA^*(\lambda))-A^*(\lambda)\rho\} \frac{1}{d_R}$$

(viii) Mean Queue Size

Let L denote the mean queue size of the model under consideration. Then

$$L = \left[\frac{d}{dz} P(z) \right]_{z=1} = \lambda P_{0,I} [P_1'(1) \psi(1) +$$

$P_1(1) \psi'(1)]$
 where,

$$P_1'(1) = \frac{M_1''(1)(E(X)-1+\rho) + (\lambda E(X(X-1))E(S) + (\lambda E(X))^2 E(S^2))(E(X)-M_1'(1))}{+E(X(X-1))(1-M_1'(1)-\rho) + 2M_1'(1)\rho(E(X)-M_1'(1))} \frac{1}{2(1-M_1'(1)-\rho)^2}$$

$$M_1''(1) = (1-A^*(\lambda)) [E(X(X-1)) + 2(q-1)(E(X)-1)]$$

$$E(S) = E(S_0)(1+a_0E(R_0)) + \sum_{i=1}^c r_i E(S_i)(1+a_i$$

$$E(R_i)) + \sum_{i=1}^c r_i p_i (E(V)+bE(D))$$

$$E(S^2) = E(S_0^2)(1+a_0E(R_0))^2 + a_0 E(R_0^2)E(S_0) + 2(1+a_0E(R_0))E(S_0)$$

$$\left[\sum_{i=1}^c r_i E(S_i)(1+a_i E(R_i)) + \sum_{i=0}^c p_i r_i (E(V)+bE(D)) \right] + \sum_{i=1}^c r_i (a_i E(R_i^2)E(S_i) + (1+a_i E(R_i))^2$$

$$E(S_i^2)) + 2(E(V)+bE(D)) \sum_{i=1}^c p_i r_i (1+a_i E(R_i))E(S_i)$$

$$+ \sum_{i=0}^c p_i r_i [E(V^2)+2bE(V)E(D) + bE(D)^2]$$

$$\psi(1) = E(S) + \frac{(1-A^*(\lambda))}{\lambda}$$

$$\psi'(1) = \lambda E(X) \left[\frac{E(S^2)}{2} + \left(\frac{1-A^*(\lambda)}{\lambda} \right) E(S) \right]$$

3 PARTICULAR CASES

Under the following conditions, the partial generating functions of the queue size at different states for the present model is given and compared with the results of Wang and Li (2008).

- The probability (b) that the server selects the setup operation is 1.
- The probability that the server takes vacation soon after each service is $p_i = 1 - \theta, \forall 0 \leq i \leq c$.
- The customers arrive in units following Poisson process i.e., $X(z) = z$.

It is found that all the partial probability generating functions exactly coincide with the corresponding results of Wang and Li (2008), under the stability condition.

4 CONCLUSION

In this paper, a more general repairable $M^X/G/1$ queue with general retrial time, Bernoulli schedule vacation policy, setup time and second multioptional services allowing reneging of customers in the retrial queue is analyzed and the system generalizes previous studies. In future, this system may be investigated under working vacation policy.

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REFERENCES

- [1] Choudhury, G., “Steady state analysis of an $M/G/1$ queue with linear retrial policy and two phase service under Bernoulli vacation schedule”, *Appl. Math. Modelling*, 32, 2480-2489, 2008.
- [2] Choudhury, G. and Deka, K., “An $M/G/1$ retrial queue with two phases of service subject to the server breakdowns and repairs”, *Quality Technology and Quantitative Management*, 65, 714-724, 2009.
- [3] Falin, G.I. and Templeton, J.G.C., “Retrial queues”, Chapman and Hall, London, 1997.
- [4] Krishnakumar, B., Vijayakumar, A. and Arivudainambi, D., “An $M/G/1$ retrial queueing system with two phase service and preemptive resume”, *Annls. of Oper. Res.*, 113, 61-79, 2002.
- [5] Li, J. and Wang, J., “An $M/G/1$ retrial queue with second multi-optional service, Feedback and unreliable server”, *Appl. Math. J. Chinese Univ. Ser. B.*, 21 (3), 252-262, 2006.
- [6] Senthilkumar, M. and Arumuganathan, R., “On the single server batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogenous service”, *Quality Technology and Quantitative Management*, 5, 145-160, 2008.
- [7] Senthilkumar, M. and Arumuganathan, R., “Performance analysis of single server retrial queue with general retrial time, impatient subscribers, two phases of service and Bernoulli schedule”, *Tamkang Journal of Science and Engg.*, 13 (2), 135-143, 2010.
- [8] Wang, J. and Li, J., “A repairable $M/G/1$ retrial queue with Bernoulli vacation and two phase service”, *Quality Technology and Quantitative Management*, 5 (2), 179-192, 2008.

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