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Common Fixed Point for Weakly Compatible Generalized φ-weak Contractive Mappings

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Abstract — In this paper we prove a common fixed point theorem for multi-valued and single-valued mappings in a metric space. The theorem use weakly compatibility and φ-weak contractivity condition with no continuity requirements. The theorem extends previous results of Rouhani and Moradi [10].

Keyword — Complete metric spaces, fixed point, generalized φ-weak contractions, weakly compatible mappings.

1. Introduction

During the last five decades, study of common fixed point of mappings satisfying contractive type conditions has been a very active field, for many mathematicians.
After Jungck has introduced the notion of compatible and weakly compatible mappings for single valued maps [6] many authors have immediately extended the concepts to multi-valued maps ([4], [5], [8]). The concepts have been used extensively to prove common fixed point theorems.
The concept of weakly contractive were defined by Daffer and Kaneko [3] in 1995. Alber and Guerr-Delabriere [2] in 1997 published some principle of weakly contractive maps in Hilbert spaces. Rhoades showed that most results of [2] are still true for any Banach spaces. Also Bae [1] obtained fixed point theorems of multi-valued weakly contractive mapping. Kammran [7], Zhang and Song [12], Beg and Abbas, Bose and Roychowdhury considered some generalized versions of these mappings and proved some fixed point theorems in complete metric space.
The purpose of the present paper is to contribute in this field of investigation by proving some fixed point theorems for weakly compatible maps with no continuity and using generalized φ-weak contractivity condition on metric spaces.

2. Preliminaries.

For convenience we start with the following definitions, lemmas, and theorems.
Let (X,d) be a metric space. We denote the family of all nonempty, bounded subset of X by $B(X)$.

Definition 2.1. Let $A,\ B\in B(X)$, then

$$H(A,B)=\max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\}$$

$$d(A,B)=\inf\{d(x,y):x\in A, y\in B\}$$

$$\delta(A,B)=\sup\{d(x,y):x\in A, y\in B\}$$

Definition 2.2. A sequence $\{A_n\}$ of nonempty subset of X is said to be convergent towards a subset A of X if
i) each point $a\in A$ is a limit of a convergent sequence $\{a_n\}$, $a_n\in A_n$ for $n\in N$.
ii) for arbitrary $\varepsilon>0$, there is an integer m such that for $n>m$, $A_n\subseteq A_\varepsilon$,
where $A_\varepsilon=\{x\in X:\exists a\in A / d(x,a)<\varepsilon\}$,
A is then said to be the limit of the sequence $\{A_n\}$.

Lemma 2.1. Let $\{A_n\}$, $\{B_n\}$ be sequences in $B(X)$ converging respectively to A and B in $B(X)$. Then the sequences of numbers $\{H(A_n,B_n)\}$, $\{d(A_n,B_n)\}$ and $\{\delta(A_n,B_n)\}$ converge to $H(A,B)$, $d(A,B)$ and $\delta(A,B)$, respectively.

Lemma 2.2. Let $\{A_n\}$ be sequences in $B(X)$ and $y\in X$ such that $\delta(A_n,y)\to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.

Definition 2.3. A mapping $T:X\to X$ is said to be φ-weak contraction if there exist a map $\varphi:[0,+\infty[\to[0,+\infty[$ with $\varphi(t)>0$ for all $t>0$ and $\varphi(0)=0$, such that

$$d(Tx,Ty)\le d(x,y)-\varphi(d(x,y)) \text{ for all } x,y\in X.$$

Definition 2.4. Two set-valued mappings $F,G:X\to B(X)$, are called generalized φ-weak contractions if there exist a map $\varphi:[0,+\infty[\to[0,+\infty[$ with $\varphi(t)>0$ for all t>0 and $\varphi(0)=0$, such that

$$H(Fx,Gy)\le M(x,y)-\varphi(M(x,y)) \text{ for all } x,y\in X,$$

where

$$M(x,y)=\max\left\{\begin{matrix} d(x,y),\ d(x,Fx),\ d(y,Gy), \\ \dfrac{d(x,Gy)+d(y,Fx)}{2}\end{matrix}\right\}.$$

Rouhani and Moradi [10] proved the following theorems that extended Nadler [9] and Daffer-Koneko's theorems [3] and Rhoades [11] and Zhang-Song's theorems [12].

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Theorem 2.1. Let (X,d) be a complete metric space and let $T,S:X\to CB(X)$ be two multi-valued mappings such that for all $x,y\in X$

$$H(Tx,Sy)\le \alpha M(x,y)$$

where $0\le\alpha<1$. Then there exists a point $x\in X$, such that $x\in Tx$ and $x\in Sx$. Moreover, if either T or S is single valued, then this common fixed point is unique.

Theorem 2.2. Let (X,d) be a complete metric space and let $T:X\to X$ and $S:X\to CB(X)$ be two mappings such that for all $x,y\in X$

$$H(\{Tx\},Sy)\le M(x,y)-\varphi(M(x,y))$$

where $\varphi:[0,+\infty[\to[0,+\infty[$ is an l.s.c. function with $\varphi(t)>0$ for all $t>0$ and $\varphi(0)=0$. Then there exists an unique point $x\in X$, such that $x=Tx\in Sx$.

Definition 2.5. [6] The self maps f, g of X are compatible iff $\lim_{n\to\infty} d(fgx_n, gfx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=t$ for some $t\in X$.

Several authors ([4], [5], [8]) extend the Definition 2.3 by introducing the compatibility of set-valued mappings as below.

Definition 2.6. The mappings $F:X\to B(X)$ and $f:X\to X$ are δ-compatible iff $\lim_{n\to\infty}\delta(fFx_n, Ffx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $fFx_n\in B(X)$, $\lim_{n\to\infty} Fx_n=\{t\}$ and $\lim_{n\to\infty} fx_n=t$ for some $t\in X$.

Definition 2.7 [6] The mappings $F:X\to B(X)$ and $f:X\to X$ are weakly compatible (or sub compatible) if they commute at coincidence points that is $\{t\in X\ /\ Ft=\{ft\}\}\subseteq\{t\in X\ /\ Fft=fFt\}$.

In [6] it is proved that weakly commuting maps are compatible but converse is not true. In [4] and [5] it is proved that δ-compatible mappings are weakly compatible but the converse is not true.

The aim of this work is to prove a theorem on the common fixed point for two multi-valued and two single valued mappings not necessary continuous, where the multi-valued mappings are generalized φ-weak contractions.

3. MAIN REZULTS.

Theorem 3.1. Let $F,G:X\to B(X)$ be mappings, where (X,d) is a complete metric space and let $I,J:X\to X$ be self mappings. Suppose that

(1) $H(Fx,Gy)\le N(x,y)-\varphi(N(x,y))$, where

$$N(x,y)=\max\left\{\begin{matrix} d(Ix,Jy), d(Ix,Fx), d(Jy,Gy), \\ \dfrac{d(Ix,Gy)+d(Jy,Fx)}{2}\end{matrix}\right\}$$

for all $x,y\in X$ and $\varphi:[0,+\infty[\to[0,+\infty[$ is an l.s.c. function with $\varphi(t)>0$ for all $t>0$ and $\varphi(0)=0$.

(2) $\cup G(X)\subseteq I(X)$, $\cup F(X)\subseteq J(X)$ and either $I(X)$ or $J(X)$ is closed

(3) the pairs of mappings $\{F, I\}$ and $\{G, J\}$ are weakly compatible,

then F, G, I, J have a unique common fixed point $u\in X$. Moreover, $Fu = Gu = \{u\}$.

Proof. Let x_0 be an arbitrary point of X. By the condition (2) there exists $x_1\in X$ such that $Jx_1\in Fx_0$. Furthermore, for this point x_1 we can choose $x_2\in X$ such that $Ix_2\in Gx_1$ and $d(Jx_1, Ix_2)\le H(Fx_0, Gx_1)$

By the condition (1) of Theorem 3.1 we have $d(Jx_1,Ix_2)\le H(Fx_0,Gx_1)\le N(x_0,x_1)-\varphi(N(x_0,x_1))$.

Continuing this process, we can define inductively the sequence $\{x_n\}$ as follows:

$Jx_{2n+1}\in Fx_{2n}$, $\quad Ix_{2n}\in Gx_{2n-1}$ for $n\in N$ and

$$\begin{aligned} d(Ix_{2n},Jx_{2n+1}) &\le H(Fx_{2n},Gx_{2n-1}) \\ &\le N(x_{2n-1},x_{2n})-\varphi(N(x_{2n-1},x_{2n})) \end{aligned}, \qquad (4)$$

where

$$d(Ix_{2n},Jx_{2n+1})\le N(x_{2n-1},x_{2n})$$

$$=\max\left\{\begin{matrix} d(Ix_{2n},Jx_{2n-1}), d(Ix_{2n},Fx_{2n}), d(Jx_{2n-1},Gx_{2n-1}), \\ \dfrac{d(Ix_{2n},Gx_{2n-1})+d(Jx_{2n-1},Fx_{2n})}{2}\end{matrix}\right\}$$

$$\le\max\left\{\begin{matrix} d(Ix_{2n},Jx_{2n-1}), d(Ix_{2n},Jx_{2n+1}), d(Jx_{2n-1},Ix_{2n}), \\ \dfrac{d(Ix_{2n},Ix_{2n})+d(Jx_{2n-1},Jx_{2n+1})}{2}\end{matrix}\right\}$$

$$\le\max\left\{\begin{matrix} d(Ix_{2n},Jx_{2n-1}), d(Ix_{2n},Jx_{2n+1}), \\ \dfrac{d(Jx_{2n-1},Ix_{2n})+d(Ix_{2n},Jx_{2n+1})}{2}\end{matrix}\right\}$$

$$\le\max\{d(Ix_{2n},Jx_{2n-1}), d(Ix_{2n},Jx_{2n+1})\}.$$

So,

$$d(Ix_{2n},Jx_{2n+1})\le N(x_{2n-1},x_{2n})\le d(Ix_{2n},Jx_{2n-1}) \qquad (5)$$

Also

$$\begin{aligned} d(Ix_{2n+2},Jx_{2n+1}) &\le H(Fx_{2n+2},Gx_{2n+1}) \\ &\le N(x_{2n},x_{2n+1})-\varphi(N(x_{2n},x_{2n+1})) \end{aligned}, \qquad (6)$$

where

$$d(Ix_{2n+2},Jx_{2n+1})\le N(x_{2n},x_{2n+1})$$

$$=\max\left\{\begin{matrix} d(Ix_{2n},Jx_{2n+1}), d(Ix_{2n},Fx_{2n}), d(Jx_{2n+1},Gx_{2n+1}), \\ \dfrac{d(Ix_{2n},Gx_{2n+1})+d(Jx_{2n+1},Fx_{2n})}{2}\end{matrix}\right\}$$

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$$\le \max\left\{\begin{matrix} d(Ix_{2n},Jx_{2n+1}), d(Ix_{2n},Jx_{2n+1}), d(Jx_{2n+1},Ix_{2n+2}), \\ \dfrac{d(Ix_{2n},Ix_{2n+2})+d(Jx_{2n+1},Jx_{2n+1})}{2} \end{matrix}\right\}$$

$$\le \max\left\{\begin{matrix} d(Ix_{2n},Jx_{2n+1}), d(Jx_{2n+1},Ix_{2n+2}), \\ \dfrac{d(Ix_{2n},Jx_{2n+1})+d(Jx_{2n+1},Ix_{2n+2})}{2} \end{matrix}\right\}$$

$$\le \max\left\{d(Ix_{2n},Jx_{2n+1}), d(Jx_{2n+1},Ix_{2n+2})\right\}.$$

So,

$$d\left(Ix_{2n+2},Jx_{2n+1}\right) \le N(x_{2n},x_{2n+2}) \le d(Ix_{2n},Jx_{2n+1}) \quad (7)$$

Let put for convenience,

$$y_k = \begin{cases} Ix_{2n} & k=2n \\ Jx_{2n-1} & k=2n-1 \end{cases}.$$

By (5) and (7), we conclude that $d\left(y_{k+1},y_k\right) \le N(y_k,y_{k-1}) \le d(y_k,y_{k-1})$, for $k \in N$.

It implies that the sequence $\left\{d\left(y_k,y_{k+1}\right)\right\}$ is monotone non-increasing and bounded below. So there exists $l \ge 0$ such that

$$\lim_{k\to\infty} d(y_k,y_{k+1}) = \lim_{k\to\infty} N(y_k,y_{k+1}) = l .$$

Since φ is lower semi-continuous,

$$\varphi(l) \le \lim_{k\to\infty}\inf \varphi(N(y_k,y_{k+1})) .$$

By (4) and (6) we conclude that

$l \le l - \varphi(l)$ i.e. $\varphi(l) = 0$ and $l = 0$, by the property of the function φ and so

$$\lim_{k\to\infty} d(y_k,y_{k+1}) = \lim_{k\to\infty} N(y_k,y_{k+1}) = 0 . \quad (8)$$

Next we show that $\{y_k\}$ is a Cauchy sequence. Let, $C_k = \sup\{d(y_i,y_j) : i, j > k\}$.

Obviously $\{C_k\}$ is decreasing. So there exists $C \ge 0$ such that $\lim_{k\to\infty} C_k = C$.

For every $p \in N$, there exist $k(p)$, $s(p) \in N$ such that $k(p)$, $s(p) \ge p$ and

$$C_p - \frac{1}{p} \le d(y_{k(p)},y_{s(p)}) \le C_p .$$

So, $\lim_{p\to\infty} d(y_{k(p)},y_{s(p)}) = C$.

Let us to prove that $C = 0$.

***Case* 1**: If $k(p)$ is even and $s(p)$ is odd, so $k(p) = 2q$ and $s(p) = 2t - 1$, by (1), we have

$$\begin{aligned} &d(y_{k(p)+1},y_{s(p)+1}) = d(y_{2q+1},y_{2t}) \le d(Jx_{2q+1},Ix_{2t}) \\ &\le H(Fx_{2q},Gx_{2t-1}) \le N(x_{2q},x_{2t-1}) - \varphi(N(x_{2q},x_{2t-1})) \end{aligned} \quad (9)$$

where,

$$d(y_{k(p)+1},y_{s(p)+1}) \le N(y_{k(p)},y_{s(p)}) = N(x_{2q},x_{2t-1})$$

$$= \max\left\{\begin{matrix} d(Ix_{2q},Jx_{2t-1}), d(Ix_{2q},Fx_{2q}), d(Jx_{2t-1},Gx_{2t-1}), \\ \dfrac{d(Ix_{2q},Gx_{2t-1})+d(Jx_{2t-1},Fx_{2q})}{2} \end{matrix}\right\}$$

$$\le \max\left\{\begin{matrix} d(Ix_{2q},Jx_{2t-1}), d(Ix_{2q},Jx_{2q+1}), d(Jx_{2t-1},Ix_{2t}), \\ \dfrac{d(Ix_{2q},Ix_{2t})+d(Jx_{2t-1},Jx_{2q+1})}{2} \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} d(y_{k(p)},y_{s(p)}), d(y_{k(p)},y_{k(p)+1}), d(y_{s(p)},y_{s(p)+1}), \\ \dfrac{d(y_{k(p)},y_{s(p)+1})+d(y_{s(p)},y_{k(p)+1})}{2} \end{matrix}\right\} \quad (10)$$

As $p \to +\infty$ in inequality (10), we have $\lim_{p\to\infty} N(y_{k(p)},y_{s(p)}) = C$. (11)

Since φ is lower semi-continuous and (8), (9), and (11) hold, we have $C \le C - \varphi(C)$. Hence $\varphi(C) = 0$ and so $C = 0$.

Similarly, if $k(p)$ is odd and $s(p)$ is even.

***Case* 2:** If $k(p)$ and $s(p)$ is even, so $k(p) = 2q$ and $s(p) = 2t$, by (1), we have

$$\begin{aligned} &d(y_{k(p)+1},y_{s(p)+1}) = d(Jx_{2q+1},Jy_{2t+1}) \le H(Fx_{2q},Fx_{2t}) \\ &\le H(Fx_{2q},Gx_{2q+1}) + H(Gx_{2q+1},Fx_{2t}) \\ &\le N(x_{2q},x_{2q+1}) - \varphi(N(x_{2q},x_{2q+1})) \\ &+N(x_{2q+1},x_{2t}) - \varphi(N(x_{2q+1},x_{2t})) \end{aligned}, \quad (12)$$

By (8), (9), (11), and (12), we have C=0.

Similarly, if $k(p)$ and $s(p)$ is odd.

Therefore, $\{y_k\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $u \in X$ such that $\lim_{k\to\infty} y_k = u$. This implies $\lim_{n\to\infty} Ix_{2n} = \lim_{n\to\infty} Jx_{2n-1} = u$ and sequences $\left\{Fx_{2n}\right\}, \left\{Gx_{2n+1}\right\}$ converge at $\left\{u\right\}$. (13)

Suppose now that the set $J(X)$ is closed. Then there is, by condition (2), a point $v \in X$ such that $Jv = u$.

Using (1) we obtain

$$H(Fx_{2n},Gv) \le N(x_{2n},v) - \varphi(N(x_{2n},v)) . \quad (14)$$

$$N(x_{2n},v) = \max\left\{\begin{matrix} d(Ix_{2n},Jv), d(Ix_{2n},Fx_{2n}), d(Jv,Gv), \\ \dfrac{d(Ix_{2n},Gv)+d(Jv,Fx_{2n})}{2} \end{matrix}\right\}.$$

Letting $n \to \infty$ in the above inequality and by (13) we get

$$\lim_{n\to\infty} N(x_{2n},v) = \max\left\{\begin{matrix} d(u,u), d(u,\{u\}), d(u,Gv), \\ \dfrac{d(u,\{u\})+d(u,Gv)}{2} \end{matrix}\right\} = d(u,Gv). \quad (15)$$

Letting $n \to \infty$ in (14) and by (15) we get $H(u,Gv) \le d(u,Gv) - \varphi(d(u,Gv))$, so $\varphi(d(u,Gv)) = 0$ and $d(u,Gv) = 0$.

Hence $Gv = \{u\} = \{Jv\}$. But the pair of maps $\{G, J\}$ are weakly compatible, thus

$GJv = JGv$, i.e. $Gu = \{Ju\}$. (16)

Now we show that u is a fixed point for G and for J. Suppose not.

By (1) we have

$$H(Fx_{2n},Gu) \le N(x_{2n},u) - \varphi(N(x_{2n},u)) \quad (17)$$

and

$$N(x_{2n},u) = \max\left\{\begin{matrix} d(Ix_{2n},Ju), d(Ix_{2n},Fx_{2n}), d(Ju,Gu), \\ \frac{d(Ix_{2n},Gu)+d(Ju,Fx_{2n})}{2} \end{matrix}\right\}.$$

Letting $n \to \infty$ in the above inequality and by (13), (16) we get

$$\lim_{n\to\infty} N(x_{2n},u) = \max\left\{\begin{matrix} d(u,Gu), d(u,\{u\}), d(Ju,\{Ju\}), \\ \frac{d(u,Gu)+d(u,Gu)}{2} \end{matrix}\right\} = d(u,Gu) \quad .(18)$$

Letting $n \to \infty$ in (17) and by (18) we get $H(u,Gu) \le d(u,Gu) - \varphi(d(u,Gu))$, so $\varphi(d(u,Gu)) = 0$ and $d(u,Gu) = 0$.

Thus we have $Gu = \{u\} = \{Ju\}$ and u is a fixed point for G and J.

Now, $\cup G(X) \subseteq I(X)$ implies that exists $w \in X$ such that $Gu = \{Iw\}$. Hence $\{u\} = Gu = \{Ju\} = \{Iw\}$.

Using (1) we get

$$H(Fw,u) = H(Fw,Gu) \le N(w,u) - \varphi(N(w,u))$$

and

$$N(w,u) = \max\left\{\begin{matrix} d(Iw,Ju), d(Iw,Fw), d(Ju,Gu), \\ \frac{d(Iw,Gu)+d(Ju,Fw)}{2} \end{matrix}\right\} = d(u,Fw).$$

So, $H(Fw,u) \le d(Fw,u) - \varphi(d(Fw,u))$ and $\varphi(d(Fw,u)) = 0$ and $d(Fw,u) = 0$ and $\{u\} = Fw = \{Iw\}$.

Hence,

$$\{u\} = Gu = \{Ju\} = Fw = \{Iw\}.$$

But the pair of maps {F,I} are weakly compatible, thus $FIw = IFw$, i.e. $Fu = \{Iu\}$.

Moreover,

$$H(Fu,u) = H(Fu,Gu) \le N(u,u) - \varphi(N(u,u))$$

and

$$N(u,u) = \max\left\{\begin{matrix} d(Iu,Ju), d(Iu,Fu), d(Ju,Gu), \\ \frac{d(Iu,Gu)+d(Ju,Fu)}{2} \end{matrix}\right\} = d(u,Fu).$$

So, $H(Fu,u) \le d(Fu,u) - \varphi(d(Fu,u))$ and $\varphi(d(Fu,u)) = 0$ and $d(Fu,u) = 0$ and $\{u\} = Gu = \{Ju\} = Fu = \{Iu\}$ i.e. u is e common fixed point for *F, G, I, J*.

Similarly, one can reach the above fact by assuming that $I(X)$ is closed.

Furthermore, we can prove that u is unique. In fact, if u and v are two common fixed points for *F, G, I, J,* then by (1) we have

$$H(u,v) = H(Fu,Gv) \le N(u,v) - \varphi(N(u,v))$$

and

$$N(u,v) = \max\left\{\begin{matrix} d(Iu,Jv), d(Iu,Fu), d(Jv,Gv), \\ \frac{d(Iu,Gv)+d(Jv,Fu)}{2} \end{matrix}\right\} = d(u,v)$$

So, $H(u,v) \le d(u,v) - \varphi(d(u,v))$ and $\varphi(d(u,v)) = 0$ and $d(u,v) = 0$ and $u=v$. The proof is complete.

Theorem 3.2. Let (X,d) be a complete metric space, and let $F,G : X \to B(X)$ be two multi-valued mappings such that for all $x, y \in X$

(1) $H(Fx,Gy) \le M(x,y) - \varphi(M(x,y)$

where $\varphi : [0,+\infty[\to [0,+\infty[$ is a l.s.c. function with $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$.

then *F, G,* have a common fixed point $u \in X$.

Proof. By taking $I = J = I_X$ we have $\cup G(X) \subseteq I(X) = X$, $\cup F(X) \subseteq J(X) = X$ and X is closed. Hence *F, G, I, J* complete the condition in Theorem 3.1 and *F, G,* have a common fixed point $u \in X$.

Remark. This theorem extends the Rohuani and Moradi theorem (Theorem 3.1. [10]).

Theorem 3.3. ([10] Theorem 4.2) Let (X,d) be a complete metric space, and let $S : X \to B(X)$ and $T : X \to X$ be two mappings such that for all $x, y \in X$

(1) $H(\{Tx\},Sy) \le M(x,y) - \varphi(M(x,y))$,

where $\varphi : [0,+\infty[\to [0,+\infty[$ is an l.s.c. function with $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$.

Then *F, I* have a unique common fixed point $u \in X$.

Proof. The proof of this theorem is similar to the Theorem 3.1 for $Gx = \{Tx\}$, $F=S$ and $I = J = I_X$.

Theorem 3.4. Let (X,d) be a complete metric spaces and *F, G, I, J:* $X \to X$ are mappings, such that for all $x, y \in X$

(1) $H(Fx,Gy) \le N(x,y) - \varphi(N(x,y))$

(2) $F(X) \subset J(X)$, $G(X) \subset I(X)$

(3) The pairs (I, F) and (J, G) are weakly compatible.

Then *F, G, I* and *J* have a unique fixed point in *X*.

Proof. The proof of this theorem is similar to the Theorem 3.1 for $Fx = \{Fx\}$ and $Gx = \{Gx\}$.

Remark. Theorem 3.4 extends the Zhang-Song theorem (Theorem 2.1. [12]).

Example. 3.4. Let X = [0, 1] endowed with metric

$$d(x,y) = \begin{cases} max\{x,y\} & x \ne y \\ 0 & x=y \end{cases}.$$

Let $F, G : X \to B(X)$ be defined by $Fx = [0, \frac{x}{4}]$, $Gy = \{\frac{y}{4}\}$ and let $I, J : X \to X$ be defined by $Ix = \begin{cases} x & x \le \frac{1}{2} \\ 2x & x > \frac{1}{2} \end{cases}$, $Jy = y$

$H(Fx, Gy) = max\{\frac{x}{4}, \frac{y}{4}\}$, $d(Jy, Gy) = y = d(Jy, Fx)$,

$$d(Ix, Jy) = \begin{cases} \max\{x, y\} & x \le \frac{1}{2} \\ \max\{2x, y\} & x > \frac{1}{2} \end{cases},$$

$$d(Ix, Fx) = \begin{cases} x & x \le \frac{1}{2} \\ 2x & x > \frac{1}{2} \end{cases},$$

$$d(Ix, Gy) = \begin{cases} \max\{x, \frac{y}{4}\} & x \le \frac{1}{2} \\ \max\{2x, \frac{y}{4}\} & x > \frac{1}{2} \end{cases},$$

$$N(x, y) = \begin{cases} x & x \le \frac{1}{2}, y \le x \\ y & x \le \frac{1}{2}, y > x \\ 2x & x > \frac{1}{2}, y \le 2x \\ y & x > \frac{1}{2}, y > 2x \end{cases}.$$

For $\varphi(t) = \begin{cases} \frac{3}{2}t^2 & t \le \frac{1}{2} \\ \frac{13}{16}t & t > \frac{1}{2} \end{cases}$

we conclude that $H(Fx, Gy) \le N(x, y) - \varphi(N(x, y))$ for all $x, y \in X$.

So *F*, *G*, *I*, *J* have a unique common fixed point ($x = 0$).

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